TITLE:

An example of random snakes by Le Gall and its applications (Recent Trends in Stochastic Models arising in Natural Phenomena and the Theory of Measure-valued Stochastic Processes)

AUTHOR(S):

Watanabe, Shinzo

CITATION:

Watanabe, Shinzo. An example of random snakes by Le Gall and its applications (Recent Trends in Stochastic Models arising in Natural Phenomena and the Theory of Measure-valued Stochastic Processes). 数理解析研究所講究録 2000, 1157: 1-16

ISSUE DATE:

2000-05

URL:

http://hdl.handle.net/2433/64181

数理解析研究所講究録
1157 巻 2000 年 1-16

An example of random snakes by Le Gall and its applications

渡辺 信三 Shinzo Watanabe, Kyoto University

1 Introduction

The notion of random snakes has been introduced by Le Gall ([Le 1], [Le 2]) to construct a class of measure-valued branching processes, called *superprocesses* or *continuous state branching processes* ([Da], [Dy]). A main idea is to produce the branching mechanism in a superprocess from a branching tree embedded in excursions at each different level of a Brownian sample path. There is no clear notion of particles in a superprocess; it is something like a *cloud* or *mist.* Nevertheless, a random snake could provide us with a clear picture of historical or genealogical developments of "particles" in a superprocess.

In this note, we give a sample pathwise construction of a random snake in the case when the underlying Markov process is a Markov chain on a tree. A simplest case has been discussed in [War 1] and [Wat 2]. The construction can be reduced to this case locally and we need to consider a recurrence family of stochastic differential equations for reflecting Brownian motions with *sticky* boundaries. A special case has been already discussed by J. Warren [War 2] with an application to a coalescing stochastic flow of piece-wise linear transformations in connection with a *non-white* or *non-Gaussian* predictable noise in the sense of B. Tsirelson.

2 Brownian snakes

Throughout this section, let $\xi = \{\xi(t), P_x\}$ be a Hunt Markov process on a locally compact separable metric space S endowed with a metric $d_S(\cdot, *)$. In examples given in later sections, we mainly consider the case when ξ is a continuous time Markov chain on a tree, however. We denote by $\mathbf{D}([0,\infty) \to S)$ ($\mathbf{D}([0,u] \to S)$) the Skorohod space formed of all right-continuous paths $w : [0,\infty)$ (resp. $[0,u]$) $\to$ S with left-hand limits (call them simply *cadlag-paths*) endowed with a Skorohod metric $d(w,w')$ and $d_u(w,w')$, respectively (cf. [B]).

In this section, we recall the notion of Brownian ξ-snake due to Le Gall ([Le 1], [Le 2]). It is defined as a diffusion process with values *in the space of cadlag stopped paths* in S so that we introduce, first of all, the following notations for several spaces of cadlag paths in S and cadlag *stopped paths* in S:

(i) for $x \in S$, $W_x(S) = \{w \in \mathbf{D}([0,\infty) \to S) \mid w(0) = x\}$,

(ii) $W(S) = \bigcup_{x\in S} W_x(S)$,

(iii) for $x \in S$,

$$\mathbf{W}_x^{stop}(S) = \{\mathbf{w} = (w,t) \mid t \in [0,\infty),\ w \in W_x(S) \text{ such that } w(s) \equiv w(s \wedge t)\ \},$$

(iv) $\mathbf{W}^{stop}(S) = \bigcup_{x\in M} \mathbf{W}_x^{stop}(S)$.

For $\mathbf{w} = (w,t) \in \mathbf{W}^{stop}(M)$, we set $\zeta(\mathbf{w}) = t$ and call it the *lifetime* of $\mathbf{w}$. Thus we may think of $\mathbf{w} \in \mathbf{W}^{stop}(S)$ a cadlag path on S stopped at its own lifetime $\zeta(\mathbf{w})$. We endow a metric on $\mathbf{W}^{stop}(S)$ by

$$d(\mathbf{w}_1, \mathbf{w}_2) = d_S(w_1(0), w_2(0)) + |\zeta(\mathbf{w}_1) - \zeta(\mathbf{w}_2)| + \int_0^{\zeta(\mathbf{w}_1)\wedge\zeta(\mathbf{w}_2)} d_u(w_1^{[u]},, w_2^{[u]})du$$

where $w^{[u]}$ is the restriction of $w \in W(S)$ on the time interval $[0,u]$. Then, $\mathbf{W}^{stop}(S)$ is a Polish space and so is also $\mathbf{W}_x^{stop}(S)$ as its closed subspace (cf. [BLL]).

2.1 Snakes with deterministic lifetimes

Let x be given and fixed. For each $0 \le a \le b$ and $\mathbf{w} = (w, \zeta(\mathbf{w})) \in \mathbf{W}_x^{stop}(S)$ such that $a \le \zeta(\mathbf{w})$, define a Borel probability $Q_{a,b}^{\mathbf{w}}(d\mathbf{w}')$ on $\mathbf{W}_x^{stop}(S)$ by the following property:

(i) $\zeta(\mathbf{w}') = b$ for $Q_{a,b}^{\mathbf{w}}$-a.a. $\mathbf{w}'$,

(ii) $w'(s) = w(s), s \in [0,a]$, for $Q_{a,b}^{\mathbf{w}}$-a.a. $\mathbf{w}'$,

(iii) under $Q_{a,b}^{\mathbf{w}}$, the shifted path $\{(w')_a^+(s) = w'(a+s), s \ge 0\}$ is equally distributed as the stopped path $\{\xi(s \wedge (b-a)), s \ge 0\}$ under $P_{w(a)}$.

Let $\zeta(t)$ be a nonnegative continuous function of $t \in [0,\infty)$ such that $\zeta(0) = 0$. Define, for each $0 \le t < t'$ and $\mathbf{w} \in \mathbf{W}_x^{stop}(S)$, a Borel probability $P(t,\mathbf{w};t',d\mathbf{w}')$ on $\mathbf{W}_x^{stop}(S)$ by

$$P(t,\mathbf{w};t',d\mathbf{w}') = Q_{m^\zeta[t,t'],\zeta(t')}^{\mathbf{w}}(d\mathbf{w}') \tag{1}$$

where

$$m^\zeta[t,t'] = \min_{t\le u\le t'} \zeta(u).$$

It is easy to see that the family $\{P(t,\mathbf{w};t',d\mathbf{w}')\}$ satisfies the Chapman-Kolmogorov equation so that it defines a family of transition probabilities on $\mathbf{W}_x^{stop}(S)$. Then, by the Kolmogorov extension theorem, we can construct a time inhomogeneous Markov process $\mathbf{X} = \{\mathbf{X}^t = (X^t(\cdot), \zeta(t))\}$ on $\mathbf{W}_x^{stop}(S)$ such that $\mathbf{X}^0 = \mathbf{x}$ where $\mathbf{x}$ is the *constant path* at x: $\mathbf{x} = (\{x(\cdot) \equiv x\}, 0)$. Note that $\zeta(\mathbf{X}^t) \equiv \zeta(t)$. If $\zeta(t)$ is Hölder-continuous, then it can be shown that a continuous modification in t of $\mathbf{X}^t$ exists (cf.[Le 1], [BLL]). In the following, we always assume that $\zeta(t)$ is Hölder-continuous so that $\mathbf{X}^t$ is continuous in t, a.s..

Definition 2.1. *The $\mathbf{W}_x^{stop}(S)$-valued continuous process $\mathbf{X} = (\mathbf{X}^t)$ is called the ξ-snake starting at $x \in M$ with the lifetime function $\zeta(t)$. Its law on $C([0,\infty) \to \mathbf{W}_x^{stop}(S))$ is denoted by $\mathbf{P}_x^\zeta$.*

We can easily see that the following three properties characterize the ξ-snake starting at $x \in M$ with the lifetime function $\zeta(t)$:

(i) $\zeta(\mathbf{X}^t) \equiv \zeta(t)$ and, for each $t \in [0,\infty)$,

$$X^t : s \in [0,\infty) \mapsto X^t(s) \in S$$

is a path of ξ-process such that $X^t(0) = x$ and stopped at time $\zeta(t)$,

(ii) for each $0 \le t < t'$,

$$X^{t'}(s) = X^t(s), \quad s \in \left[0, m^{\zeta}[t,t']\right],$$

(iii) for each $0 \le t < t'$, $\{X^{t'}(s); s \ge m^{\zeta}[t,t']\}$ and $\{X^u(\cdot); u \le t\}$ are independent given $X^{t'}(m^{\zeta}[t,t'])$.

2.2 Brownian snakes

In the following, we denote by $RBM^x([0,\infty))$ a reflecting Brownian motion $R = (R(t))$ on $[0,\infty)$ with $R(0) = x$.

Definition 2.2. *The* Brownian ξ-snake $\mathbf{X} = (\mathbf{X}^t)$ starting at $x \in S$ *is a* $\mathbf{W}_x^{stop}(S)$-*valued continuous process with the law on* $C([0,\infty) \to \mathbf{W}_x^{stop}(S))$ *given by*

$$\mathbf{P}_x(\cdot) = \int_{C([0,\infty)\to[0,\infty))} \mathbf{P}_x^{\zeta}(\cdot) P^R(d\zeta) \tag{2}$$

where P^R *is the law on* $C([0,\infty) \to [0,\infty))$ *of* $RBM^0([0,\infty))$.

It is obvious that $\mathbf{X}^0 = \mathbf{x}$, a.s..

Proposition 2.1. *([Le 1], [BLL])* $\mathbf{X} = (\mathbf{X}^t)$ *is a time homogeneous diffusion on* $\mathbf{W}_x^{stop}(S)$ *with the transition probability*

$$P(t, \mathbf{w}, d\mathbf{w}') = \iint_{0 \le a \le b < \infty} \Theta_t^{\zeta(\mathbf{w})}(da, db) Q_{a,b}^{\mathbf{w}}(d\mathbf{w}') \tag{3}$$

where $\Theta_t^{\zeta(\mathbf{w})}(da, db)$ *is the joint law of* $(\min_{0 \le s \le t} R(s), R(t))$, $R(t)$ *being* $RBM^{\zeta(\mathbf{w})}([0,\infty))$; *explicitly,*

$$\begin{aligned} \Theta_t^{\zeta(\mathbf{w})}(da, db) &= \frac{2(\zeta(\mathbf{w}) + b - 2a)}{\sqrt{2\pi t^3}} e^{-\frac{(\zeta(\mathbf{w})+b-2a)^2}{2t}} 1_{\{0<a<b\wedge\zeta(\mathbf{w})\}} \\ &+ \sqrt{\frac{2}{\pi t}} e^{-\frac{(\zeta(\mathbf{w})+b)^2}{2t}} 1_{\{0<b\}} \delta_0(da) db. \end{aligned} \tag{4}$$

The lifetime process $\zeta(t) := \zeta(\mathbf{X}^t)$ *is a* $RBM^0([0,\infty))$ *and, conditioned on the process* $\zeta = (\zeta(t))$, *it is the* ξ-*snake with the deterministic lifetime fuction* $\zeta(t)$.

Remark 2.1. The term "Brownian" in a Brownian snake indicates that its branching mechanism is Brownian, that is, its lifetime process is a reflecting Brownian motion, not that its underlying Markov process is Brownian; it can be an arbitrary Markov process.

2.3 The snake description of superprocess $\{\mu(t), \mathbf{P}_\mu\}$

Let $x \in S$ and $\mathbf{X} = (\mathbf{X}^t)$ be the Brownian ξ-snake starting at x. Then $\zeta(\mathbf{X}^t)$ is a $RBM^0([0,\infty))$. Let

$$l(t,a) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[a,a+\varepsilon)}(\zeta(\mathbf{X}^s))ds \tag{5}$$

be its local time at $a \in [0,\infty)$.

Let $\mathcal{M}_F(S)$ be the space of all finite Borel measures on S with the topology of weak convergence and $C_b(S)$ be the space of all bounded continuous functions on S. Introduce the usual notation

$$\langle \mu, f \rangle = \int_S f(x)\mu(dx), \quad \mu \in \mathcal{M}_F(S),\ f \in C_b(S).$$

Let $(\mu(t), P_\mu)$ be the $(\xi, \psi(x,z) = -z^2)$-superprocess ([Da],[Dy]): It is a diffusion process on $\mathcal{M}_F(S)$ with the branching property of which the log-Laplace functional

$$u(t,x) = -\log \mathbf{E}_{\delta_x}[\exp(-\langle \mu(t), f \rangle)], \quad t > 0,\ x \in S,$$

is the solution to the initial value problem

$$\frac{\partial u}{\partial t} = Lu + \psi(\cdot, u), \quad u(0+,\cdot) = f,$$

where L is the generator of ξ. Then, for $\gamma > 0$ and $x \in S$, the process $\mu(t)$ under $P_{\gamma \cdot \delta_x}$ can be constructed from the Brownian ξ-snake $\mathbf{X} = (\mathbf{X}^t)$ starting at x in the following way: Define $\mu(t) \in \mathcal{M}_F(S), t \geq 0$, by

$$\langle \mu(t), f \rangle = \int_0^{l^{-1}(\gamma,0)} f(\langle \mathbf{X}^s \rangle) l(ds,t), \quad f \in C_b(S), \tag{6}$$

where $\langle \mathbf{X}^t \rangle = X^t(\zeta(\mathbf{X}^t)) \in S$: the position of $\mathbf{X}^t$ stopped at its lifetime $\zeta(\mathbf{X}^t)$ and

$$l^{-1}(\gamma, 0) = \inf\{u \mid l(u,0) > \gamma\}.$$

Theorem 2.1. *(Le Gall [Le 1], [Le 2])* $\{\mu(t)\}$ *defined by (6) is exactly the* $(\xi, \psi(x,z) = -z^2)$*-superprocess* $\{\mu(t)\}$ *under* $P_{\gamma \cdot \delta_x}$.

3 ξ-snake where ξ is a Markov chain on a tree

3.1 The case that ξ is trivial

The simplest case of Brownian ξ-snakes is when the state space S of the underlying motion $\xi = \{\xi_t\}$ consists of a single point: $S = \{a\}$, so that ξ is a trivial motion $\xi_t \equiv a$. In this case, the snake can be identified with its lifetime so that it is a reflecting Brownian motion $R = (R(t))$ on $[0,\infty)$ with $R(0) = 0$.

3.2 The case that ξ is a holding time process

The next simplest case was studied in [Wat 2] (cf. also [War 1]). This is the case when $S = \{a, b\}$, the state b being a trap, so that

$$\xi(t) = \begin{cases} a, & 0 \leq t < e \\ b, & t \geq e \end{cases},$$

where e is an exponential holding time with parameter θ, i.e., with mean $1/\theta$. In this case, the snake $\mathbf{X} = (\mathbf{X}^t)$ which starts at the constant path at a moves in the following subspace $\mathbf{W}$ of $\mathbf{W}_a^{stop}(S)$:

$$\mathbf{W} = \{\ \mathbf{w}_{[x,y]}\ ; \quad x = y = 0 \text{ or } 0 < x \leq y < \infty\}$$

where $\mathbf{w}_{[x,y]} \in \mathbf{W}_a^{stop}(S)$ is defined by

(i) $\mathbf{w}_{[0,0]} = \mathbf{a}$: the constant path at a,

(ii) for $x > 0$, $\mathbf{w}_{[x,x]} = (w, \zeta(\mathbf{w}_{[x,x]}))$ where $w(t) \equiv a$ and $\zeta(\mathbf{w}_{[x,x]}) = x$,

(iii) for $0 < x < y$, $\mathbf{w}_{[x,y]} = (w, \zeta(\mathbf{w}_{[x,y]}))$ where

$$w(t) = \begin{cases} a, & 0 \leq t < x \\ b, & t \geq x \end{cases}$$

and $\zeta(\mathbf{w}_{[x,y]}) = y$.

Then, $\mathbf{W} \cong D := \{(0,0)\} \cup \{\ (x,y) \mid 0 < x \leq y < \infty\ \}$ and the topology coincides with the relative topology of $\mathbf{R}^2$.

For a given constant $\theta > 0$ and a Brownian motion (B_t) on $\mathbf{R}$ with $B_0 = 0$ (denote it simply by $BM^0(\mathbf{R})$), consider the following stochastic differential equation:

$$dX_t = 1_{\{X_t > 0\}} dB_t + \frac{\theta}{2} 1_{\{X_t = 0\}} dt, \quad X_0 = x \geq 0. \tag{7}$$

Let

$$R_t = B_t + L_t, \quad L_t = -\min_{0 \leq s \leq t} B_s \tag{8}$$

so that R_t is $RBM^0([0,\infty))$ and L_t is its local time at 0 thus giving its Skorohod decomposition of R_t (cf. [IW], p.120).

Theorem 3.1. *(1) The SDE (7) has a solution $X = (X_t)$ such that $X_t \geq 0$ for all t. Furthermore, the law of the joint process (B_t, X_t) is uniquely determined.*

(2) Let (B_t, X_t) be a solution of (7) with $X_0 = 0$ and set

$$X_t^{(0)} = R_t \quad \text{and} \quad X_t^{(1)} = X_t$$

where R_t is given by (8). Then, with probability one, it holds that

$$X_t^{(1)} \leq X_t^{(0)} \quad \text{for all} \quad t \geq 0$$

and that

$$X_t^{(1)} = X_t^{(0)} \quad \text{implies} \quad X_t^{(1)} = X_t^{(0)} = 0.$$

The second part of the theorem implies that, if we set

$$x_t = X_t^{(0)} - X_t^{(1)} \quad \text{and} \quad y_t = X_t^{(0)}, \tag{9}$$

then, with probability one, $(x_t, y_t) \in D$ for all $t \geq 0$.

Theorem 3.2. *([War 1],[Wat 2]) The Brownian ξ-snake $\mathbf{X} = (\mathbf{X}^t)$ starting at a is given by*

$$\mathbf{X}^t = \mathbf{w}_{[x_t, y_t]}$$

where (x_t, y_t) is given by (9).

Proof of Theorem 3.1. The uniqueness of solutions for equation (7) can be deduced in the usual way as follows (cf. [IW]). Let (B_t, X_t) satisfy the equation (7). It is easy to see that $X_t \geq 0$ for all $t \geq 0$, a.s.. Set $A_t = \int_0^t 1_{\{X_s > 0\}} ds$ and A_t^{-1} be the right-continuous inverse of $t \to A_t$. Then

$$W_t = \int_0^{A_t^{-1}} 1_{\{X_s > 0\}} dB_s \quad \text{is} \quad BM^0(\mathbf{R}) \quad \text{and} \quad X_{A_t^{-1}} = x + W_t + \phi_t$$

where

$$\phi_t = \frac{\theta}{2} \int_0^{A_t^{-1}} 1_{\{X_s = 0\}} ds.$$

This is a Skorohod equation (cf. [IW], p.121) so that $\widetilde{X}_t := X_{A_t^{-1}}$ is $RBM^0([0, \infty))$ and ϕ_t is the local time at 0 of $\widetilde{X}_t$. $\widetilde{X}_t$ and ϕ_t are uniquely determined from W_t as $\phi_t = -\inf_{0 \leq s \leq t}(x + W_s) \wedge 0$ and $\widetilde{X}_t = W_t + \phi_t$. Since $t = A_t + \int_0^t 1_{\{X_s = 0\}} ds$, we have

$$A_t^{-1} = t + \frac{2}{\theta} \phi_t.$$

Let $a_t = \int_0^t 1_{\{X_s = 0\}} ds$. By Knight's theorem ([IW], p.86), $\widetilde{W}_t := \int_0^{a_t^{-1}} 1_{\{X_s = 0\}} dB_s$ is a $BM^0(\mathbf{R})$ which is independent of $W = (W_t)$. Then,

$$B_t = \int_0^t 1_{\{X_s > 0\}} dB_s + \int_0^t 1_{\{X_s = 0\}} dB_s = W_{A_t} + \widetilde{W}_{a_t}.$$

Also, we have

$$a_t = t - A_t = \frac{2}{\theta} \phi_{A_t}.$$

In summing up the above discussions, we can deduce that the joint process $(B(t), X_t)$ is uniquely determined from two mutually independent $BM^0(\mathbf{R})$'s $W = (W_t)$ and $\widetilde{W} = (\widetilde{W}_t)$ as follows:

$$\widetilde{X}_t = x + W_t + \phi_t, \quad A_t^{-1} = t + \frac{2}{\theta} \phi_t, \quad A_t = \text{the inverse of} \quad t \to A_t^{-1},$$

$$a_t = t - A_t = \frac{2}{\theta} \phi_{A_t}, \quad X_t = \widetilde{X}_{A_t}, \quad B_t = W_{A_t} + \widetilde{W}_{a_t}.$$

This clearly implies the uniqueness in law of the process (B_t, X_t).

Conversely, given two mutually independent $BM^0(\mathbf{R})$'s W and $\widetilde{W}$, if we define (B_t, X_t) as above, then we can show that it satisfies the equation (7). This proves the first part of the theorem.

For the proof of the second part, we consider the process (B_t, X_t) satisfying (7), $X_t \geq 0$ for $t \geq 0$ and $X_0 = 0$. Using the same notations as above, set

$$S_t = \widetilde{W}_t - \frac{\theta}{2}t.$$

Then we have

$$\begin{aligned} X_t &= \int_0^t 1_{\{X_s>0\}} dB_s + \frac{\theta}{2}\int_0^t 1_{\{X_s=0\}} ds \\ &= B_t - \int_0^t 1_{\{X_s=0\}} dB_s + \frac{\theta}{2}\int_0^t 1_{\{X_s=0\}} ds = B_t - S_{a_t}. \end{aligned}$$

Set

$$\overline{S}_t = S_t + K_t, \quad \text{where} \quad K_t = -\inf_{0\leq s\leq t} S_s,$$

so that $\overline{S}_t$ is a reflecting Brownian motion with drift $\frac{\theta}{2}t$ towards the origin. For R_t and L_t defined by (8), we have, therefore,

$$X_t = B_t - S_{a_t} = R_t - L_t - \overline{S}_{a_t} + K_{a_t}$$

and we can show that $L_t = K_{a_t}$ (cf. [War 1]) so that we have finally

$$X_t = R_t - \overline{S}_{a_t}.$$

This proves that $X_t (= X_t^{(1)}) \leq R_t (= X_t^{(0)})$.

Next we show that $X_t = R_t$ implies that $X_t = 0$. We have seen above that

$$R_t - X_t = \overline{S}_{a_t}$$

where $a_t = \int_0^t 1_{\{X_s=0\}} ds$. Note that the processes (X_t) and $(\overline{S}_t)$ are mutually independent because of the independence of W and $\widetilde{W}$. Let $\mathcal{Z} = \{\ t \mid \overline{S}_t = 0\ \}$. If $\mathcal{C} = \{\alpha_n\} \subset (0, \infty)$ is a deterministic countable set, then

$$P(\mathcal{C} \cap \mathcal{Z} \neq \emptyset) \leq \sum_n P(\alpha_n \in \mathcal{Z}) = \sum_n P(\overline{S}_{\alpha_n} = 0) = 0. \tag{10}$$

Let

$$\{\ t \in (0, \infty) \mid X_t > 0\ \} = [0, \infty) \setminus \{\ t \mid X_t = 0\ \} := \bigcup_\alpha e_\alpha$$

where $\{e_\alpha\}$ is a countable family of disjoint open intervals. Since a_t is constant $(:= \beta_\alpha)$ on each interval e_α,

$$\mathcal{D} = \{\ a_t \mid t \in \bigcup_\alpha e_\alpha\ \} = \{\beta_\alpha\}$$

is a countable set. The random sets $\mathcal{D} = \{\beta_\alpha\}$ and $\mathcal{Z}$ are mutually independent because processes (X_t) and $(\overline{S}_t)$ are mutually independent. Hence, by the Fubini theorem and (10), we have

$$P(\mathcal{D} \cap \mathcal{Z} \neq \emptyset) = 0, \quad \text{i.e.} \quad P(\mathcal{D} \cap \mathcal{Z} = \emptyset) = 1,$$

which implies that, almost surely,

$$X_t > 0 \Longrightarrow \overline{S}_{a_t} > 0, \quad \text{equivalently,} \quad R_t - X_t = 0 \Longrightarrow X_t = 0.$$

Proof of Theorem 3.2 We remarked above that ξ-snake $\mathbf{X} = (\mathbf{X}^t)$ starting at a is given by

$$\mathbf{X}^t = \mathbf{w}_{[x_t, y_t]}$$

where (x_t, y_t) is a diffusion process on D starting at $(0,0)$. If we set

$$X_t = y_t - x_t \quad \text{and} \quad R_t = y_t,$$

then (X_t, R_t) is a diffusion process on $\widetilde{D} = \{(0,0)\} \cup \{ (\lambda, \sigma) \mid 0 \le \lambda < \sigma < \infty \}$ starting at $(0,0)$ and, by (3), its transition probability is given explicitly as follows:

$$p(t, (\lambda, \sigma), d\lambda' d\sigma') = \iint_{0 \le a < b < \infty} \Theta_t^\sigma(da, db) q_{a,b}^{(\lambda,\sigma)}(d\lambda' d\sigma'),$$

where

$$\begin{aligned} & q_{a,b}^{(\lambda,\sigma)}(d\lambda' d\sigma') \\ = \; & 1_{\{\sigma - \lambda < a\}} \cdot \delta_{\sigma' - \sigma + \lambda}(d\lambda') \cdot \delta_b(d\sigma') \\ + \; & 1_{\{\sigma - \lambda \ge a\}} \cdot 1_{\{0 < \lambda' < \sigma' - a\}} \cdot \theta \cdot e^{-\theta(\sigma' - \lambda' - a)} \cdot d\lambda' \cdot \delta_b(d\sigma') \\ + \; & 1_{\{\sigma - \lambda \ge a\}} \cdot e^{-\theta(\sigma' - a)} \cdot \delta_0(d\lambda') \cdot \delta_b(d\sigma'). \end{aligned}$$

Here $\Theta_t^\sigma(da, db) = P_\sigma(\min_{0 \le s \le t} R(s) \in da, R(t) \in db)$, P_σ being the probability law governing the standard reflecting Brownian motion $R = (R(t))$ with $R(0) = \sigma$: It is given explicitly by

$$\begin{aligned} \Theta_t^\sigma(da, db) \; = \; & \frac{2(\sigma + b - 2a)}{\sqrt{2\pi t^3}} e^{-\frac{(\sigma+b-2a)^2}{2t}} 1_{\{0 < a < b \wedge \sigma\}} da db \\ & + \sqrt{\frac{2}{\pi t}} e^{-\frac{(\sigma+b)^2}{2t}} 1_{\{0 < b\}} \delta_0(da) db. \end{aligned}$$

From this explicit expression, we can prove directly that R_t is a $RBM^0([0,\infty))$ and that, if $R_t = B_t + L_t$ is the semi-martingale decomposition (indeed, the Skorohod decomposition) of R_t, then (X_t, B_t) satisfies SDE (7) (cf. [DS]).

3.3 The case that ξ is a Markov chain on a tree.

Here, we only consider a tree without terminating branches, for simplicity. By a *tree*, we mean a collection S of finite sequences $a_1 \cdots a_m$ of positive integers with the following properties:

(1) $1 \in S$.

(2) $a_1 \cdots a_m \in S \Longrightarrow a_1 = 1$.

(3) If $a_1 \cdots a_m \in S$, then there exists a positive integer $1 \leq N := N(a_1 \cdots a_m)$ such that

$$a_1 \cdots a_m a_{m+1} \in S \quad \text{if and only if} \quad 1 \leq a_{m+1} \leq N.$$

In particular, $a_1 \cdots a_m 1 \in S$.

Thus, S consists of

$$1,\ 11, 12, \ldots, 1N(1),\ 111, 112, \ldots, 11N(11),\ 121, 122, \ldots, 12N(12),\ \ldots,.$$

For $\tau = a_1 \cdots a_m \in S$, we set $A(\tau) = \{\, a_1 \cdots a_m a_{m+1} \mid 1 \leq a_{m+1} \leq N(\tau) \,\}$ and call $\eta \in A(\tau)$ a *child* of τ so that $A(\tau)$ is the set of all children of τ. $1 \in S$ is called the *root* of S.

Let a tree S be given and fixed. S is a countable set and we endow on it the discrete topology. Suppose we are given the following quantities:

(1) $\theta(\tau) > 0$ for $\tau \in S$.

(2) $\pi(\tau, \eta) > 0$ for $\tau \in S$ and $\eta \in A(\tau)$ such that

$$\sum_{\eta \in A(\tau)} \pi(\tau, \eta) = 1, \quad \forall \tau \in S.$$

Then a Hunt Markov process $\xi = (\xi_t)$ on S starting at the root 1 can be determined as follows. $\xi_0 = 1$ and stays at 1 during the exponential holding time with parameter $\theta(1)$, (i.e., with wean $1/\theta(1)$). Then it jumps to $\tau \in A(1)$ with probability $\pi(1, \tau)$. Then it stays at τ during independent exponential time with parameter $\theta(\tau)$ and then jumps to $\eta \in A(\tau)$ with probability $\pi(\tau, \eta)$, and so on.

We are interested in the Brownian ξ snake $\mathbf{X} = (\mathbf{X}_t)$ starting at the constant path at the root 1, particularly in its sample paths structure. As we shall see, the sample paths of the snake can be constructed by applying recurrently the construction given in the previous subsection.

Step 1. We construct $(X_t^{(0)}, X_t^{(1)}) \in [0, \infty)^2$ with $(X_0^{(0)}, X_0^{(1)}) = (0, 0)$ in the same way as in subsection 3.2: For a $BM^0(\mathbf{R})$ B_t,

$$X_t^{(0)} = B_t + L_t, \quad \text{where} \quad L_t = -\inf_{0 \leq s \leq t} B_s. \tag{11}$$

and

$$X_t^{(1)} = \int_0^t 1_{\{X_s^{(1)} > 0\}} dB_s + \frac{\theta(1)}{2} \int_0^t 1_{\{X_s^{(1)} = 0\}} ds. \tag{12}$$

We have seen above that the law of the joint process $(X_t^{(0)}, X_t^{(1)})$ is uniquely determined and that, with probability one,

$$X_t^{(0)} \geq X_t^{(1)}, \quad \text{and,} \quad X_t^{(0)} = X_t^{(1)} \Longrightarrow X_t^{(0)} = X_t^{(1)} = 0. \tag{13}$$

Set

$$n_t^{(0)} \equiv 1 \in S. \tag{14}$$

Step 2. For each sample path of $X_t^{(1)}$, define

$$[0,\infty) \setminus \{\, t \mid X_t^{(1)} = 0 \,\} = \bigcup_\alpha e_\alpha^{(1)}$$

where $\{e_\alpha^{(1)}\}$ is a family of disjoint open intervals. Each $e_\alpha^{(1)}$ is called an *excursion interval of $X_t^{(1)}$ away from* 0. Given the joint process $(X^{(0)}, X^{(1)}, n^{(0)})$, we set up a family $\{\rho_\alpha^{(1)}\}$ of $A(1)(\subset S)$-valued random variables, indexed by excursion intervals $\{e_\alpha^{(1)}\}$, which are mutually independent (under the conditional law $P(\,\cdot\,|X^{(0)}, X^{(1)}, n^{(0)})$) and identically distributed as

$$P(\rho_\alpha^{(1)} = \tau | X^{(0)}, X^{(1)}, n^{(0)}) = \pi(1,\tau), \quad \tau \in A(1).$$

Define $n_t^{(1)}$, $t \in [0,\infty) \setminus \{\, t \mid X_t^{(1)} = 0 \,\}$, by

$$n_t^{(1)} = \rho_\alpha^{(1)}, \quad t \in e_\alpha^{(1)}. \tag{15}$$

Thus, we have defined the joint process $(X^{(0)}, X^{(1)}, n^{(0)}, n^{(1)})$ on $[0,\infty)^2 \times S^2$. Note that $n^{(1)} = (n_t^{(1)})$ is defined only for such t that $t > 0$ and $X_t^{(1)} > 0$.

Step 3. By repeating a same argument as in subsection 3.2, we can show that there exists a joint process $(X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)})$ on $[0,\infty)^3 \times S^2$ such that

(i) The process $(X^{(0)}, X^{(1)}, n^{(0)}, n^{(1)})$ on $[0,\infty)^2 \times S^2$ has the same law as that given in Step 2.

(ii) If B_t is defined by (11), then $(X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)})$ satisfies SDE (12) and the following SDE combined together:

$$X_t^{(2)} = \int_0^t 1_{\{X_s^{(1)}>0, X_s^{(2)}>0\}} dB_s + \frac{1}{2}\int_0^t \theta(n_s^{(1)}) 1_{\{X_s^{(1)}>0, X_s^{(2)}=0\}} ds. \tag{16}$$

Furthermore, the law of the joint process $(X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)})$ is uniquely determined. Also, we can show that, with probability one,

$$X_t^{(1)} \geq X_t^{(2)}, \quad \text{and,} \quad X_t^{(1)} = X_t^{(2)} \Longrightarrow X_t^{(1)} = X_t^{(2)} = 0. \tag{17}$$

Step 4. For each sample path of $X_t^{(2)}$, define

$$[0,\infty) \setminus \{\, t \mid X_t^{(2)} = 0 \,\} = \bigcup_\beta e_\beta^{(2)}$$

where $\{e_\beta^{(2)}\}$ is a family of disjoint open intervals. Each $e_\beta^{(2)}$ is called an *excursion interval of $X_t^{(2)}$ away from* 0. Since $X_t^{(1)} \geq X_t^{(2)}$, each excursion interval $e_\beta^{(2)}$ is contained in exactly one excursion interval $e_\alpha^{(1)}$ of $X^{(1)}$. Given the joint process $(X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)})$, we set up a family $\{\rho_\beta^{(2)}\}$ of S-valued random variables, indexed by excursion intervals $\{e_\beta^{(2)}\}$ of $X^{(2)}$, which are mutually independent (under the conditional law $P(\,\cdot\,|X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)}))$ and distributed as

$$P(\,\rho_\beta^{(2)} = \tau\,|X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)}) = \pi(\rho_\alpha^{(1)}, \tau), \quad \tau \in A(\rho_\alpha^{(1)}),$$

(α is determined by the unique excursion interval $e_\alpha^{(1)}$ of $X^{(1)}$ containing $e_\beta^{(2)}$). Define $n_t^{(2)}$, $t \in [0,\infty) \setminus \{\, t \mid X_t^{(2)} = 0 \,\}$, by

$$n_t^{(2)} = \rho_\beta^{(2)}, \quad t \in e_\beta^{(2)}. \tag{18}$$

Thus, we have defined the joint process $(X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)}, n^{(2)})$ on $[0,\infty)^3 \times S^3$. Note that $n^{(2)} = (n_t^{(2)})$ is defined only for such t that $t > 0$ and $X_t^{(2)} > 0$.

Step 5. Again repeating a same argument, we can show that there exists a joint process $(X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)}, n^{(0)}, n^{(1)}, n^{(2)})$ on $[0,\infty)^4 \times S^3$ such that

(i) The process $(X^{(0)}, X^{(1)}, X^{(2)}, n^{(0)}, n^{(1)}, n^{(2)})$ on $[0,\infty)^3 \times S^3$ has the same law as that given in Step 4.

(ii) If B_t is defined by (11), then $(X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)}, n^{(0)}, n^{(1)}, n^{(2)})$ satisfies SDE (12), SDE (16) and the following, combined together:

$$X_t^{(3)} = \int_0^t 1_{\{X_s^{(1)}>0, X_s^{(2)}>0, X_s^{(3)}>0\}} dB_s + \frac{1}{2}\int_0^t \theta(n_s^{(2)}) 1_{\{X_s^{(1)}>0, X_s^{(2)}>0, X_s^{(3)}=0\}} ds. \tag{19}$$

Furthermore, the law of the joint process $(X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)}, n^{(0)}, n^{(1)}, n^{(2)})$ is uniquely determined. Also, we can show that, with probability one,

$$X_t^{(2)} \geq X_t^{(3)}, \quad \text{and,} \quad X_t^{(2)} = X_t^{(3)} \Longrightarrow X_t^{(2)} = X_t^{(3)} = 0. \tag{20}$$

We continue these steps recurrently. Then we obtain the following joint process

$$(X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)}, \cdots, n^{(0)}, n^{(1)}, n^{(2)}, n^{(3)}, \cdots).$$

$n_t^{(k)}$ is S-valued and it is defined for such t that $t > 0$ and $X_t^{(k)} > 0$. We have also that, with probability one,

$$X_t^{(0)} \geq X_t^{(1)} \geq \cdots X_t^{(k)} \geq \cdots$$

and

$$X_t^{(k)} = X_t^{(k+1)} \Longrightarrow X_t^{(k)} = X_t^{(k+1)} = \cdots = 0.$$

Hence, we have that, with probability one,

$$X_t^{(0)} > X_t^{(1)} > \cdots > X_t^{(k)} > X_t^{(k+1)} = X_t^{(l+2)} = \cdots = 0, \text{ for some } k \geq -1$$

or

$$X_t^{(0)} > X_t^{(1)} > \cdots > X_t^{(m)} > X_t^{(m+1)} > \cdots > 0.$$

Definition 3.1. *Define $N_t = k \vee 0$ in the first case and $N_t = \infty$ in the second case.*

Proposition 3.1. *For each fixed $t > 0$, $P(N_t < \infty) = 1$.*

The proof is given by Warren ([War 2]) in the case $\theta(\tau)$ is constant and it can be modified to the general case.

For given $n \geq 1$, $x_0 > 0, x_1 > 0, \ldots, x_n > 0$ and $\tau_0 = 1, \tau_1, \ldots, \tau_n \in S$ such that $\tau_k \in A(\tau_{k-1})$, $k = 1, 2, \ldots, n$, we define a path $w \in \mathbf{D}([0,\infty) \to S)$ by

$$w(t) = \begin{cases} 1, & 0 \leq t < x_0 \\ \tau_1, & x_0 \leq t < x_0 + x_1 \\ \ldots, & \ldots \\ \ldots, & \ldots \\ \tau_{n-1}, & x_0 + x_1 + \cdots + x_{n-2} \leq t < x_0 + x_1 + \cdots + x_{n-1} \\ \tau_n, & t \geq x_0 + x_1 + \cdots + x_{n-1}. \end{cases}$$

Then we define $\mathbf{w} = (w, \zeta(\mathbf{w})) \in \mathbf{W}_1^{stop}(S)$ by setting

$$\zeta(\mathbf{w}) = x_0 + x_1 + \cdots + x_n$$

and denote it by $\mathbf{w}_{\{x_0,\ldots,x_n;\tau_0,\ldots,\tau_n\}}$.

We now define, from the joint process $(X^{(0)}, X^{(1)}, \cdots, n^{(0)}, n^{(1)}, \cdots)$ constructed above, a $\mathbf{W}_1^{stop}(S)$-valued process

$$\mathbf{X} : [0,\infty) \ni t \longmapsto \mathbf{X}^t \in \mathbf{W}_1^{stop}(S)$$

in the following way:

(i) When $X_t^{(0)} = 0$, i.e., when $X_t^{(0)} = X_t^{(1)} = \cdots = 0$, we set

$$\mathbf{X}^t = \mathbf{1};\ \text{the constant path at } 1 \in S.$$

(ii) When $X_t^{(0)} > 0$ and $N_t = k$, i.e., when

$$X_t^{(0)} > X_t^{(1)} > \cdots > X_t^{(k)} > X_t^{(k+1)} = X_t^{(k+2)} = \cdots = 0,$$

we set

$$\mathbf{X}^t = \mathbf{w}_{\{X_t^{(0)} - X_t^{(1)}, X_t^{(1)} - X_t^{(2)}, \ldots, X_t^{(k)};\ n_t^{(0)}, n_t^{(1)}, \ldots, n_t^{(k)}\}}.$$

Theorem 3.3. $\mathbf{X} = (\mathbf{X}^t)$ *defines a diffusion process on $\mathbf{W}_1^{stop}(S)$ and it coinsides with the Brownian ξ-snake starting at $\mathbf{1}$.*

The proof can be reduced, locally, to that of Theorem 3.2: namely, in each excursion interval of $X^{(1)}$, for example, we can deduce that $X_t^{(2)}$ satisfies the equation (16) and so on. Details will be omitted.

4 Some applications

4.1 A theorem of the Ray-Knight type

Consider a simple case of $S = \{1, 11, 111, \ldots\}$, i.e., $A(\tau) = \{\tau 1\}$ for all τ. We identify

$$\overbrace{1\cdots 1}^{m} \in S$$

with the integer $m-1$ so that the root 1 is now denoted by 0. Then the above joint process $(X^{(0)}, X^{(1)}, \cdots,)$ is uniquely determined (in the law sense) by the following system of SDE's:

$$X_t^{(0)} = B_t + L_t, \quad \text{where} \quad L_t = -\inf_{0 \le s \le t} B_s$$

and

$$\begin{aligned}
X_t^{(1)} &= \int_0^t 1_{\{X_s^{(1)}>0\}} dB_s + \frac{\theta(0)}{2} \int_0^t 1_{\{X_s^{(1)}=0\}} ds, \\
X_t^{(2)} &= \int_0^t 1_{\{X_s^{(1)}>0, X_s^{(2)}>0\}} dB_s + \frac{\theta(1)}{2} \int_0^t 1_{\{X_s^{(1)}>0, X_s^{(2)}=0\}} ds, \\
\cdots & \quad \cdots, \\
X_t^{(k)} &= \int_0^t 1_{\{X_s^{(1)}>0, \ldots, X_s^{(k-1)}>0, X_s^{(k)}>0\}} dB_s + \frac{\theta(k-1)}{2} \int_0^t 1_{\{X_s^{(1)}>0, \ldots, X_s^{(k-1)}>0, X_s^{(k)}=0\}} ds, \\
\cdots & \quad \cdots .
\end{aligned}$$

$X^{(0)}$ is a $RBM^0([0,\infty)$ and let $l(t,a)$ be its local time at a:

$$l(t,a) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[a,a+\varepsilon)}(X_s^{(0)}))ds.$$

Set, for $\gamma > 0$,

$$\mu_t^{(k)} = \int_0^{l^{-1}(\gamma,0)} 1_{\{N_s=k\}} l(ds,t) \quad t \ge 0, \quad k = 0, 1, \ldots .$$

Then we have

Theorem 4.1. *The joint process $(\mu_t^{(0)}, \mu_t^{(1)}, \cdots)$ defines a diffusion process on $[0,\infty)^\infty$ starting at $(\gamma, 0, 0, \cdots)$ uniquely determined by the strong solution of the following SDE:*

$$\begin{aligned}
d\mu_t^{(0)} &= \sqrt{2\mu_t^{(0)} \vee 0} \cdot db_t^{(0)} - \theta(0) \cdot \mu_t^{(0)} dt, \\
d\mu_t^{(1)} &= \sqrt{2\mu_t^{(1)} \vee 0} \cdot db_t^{(1)} + \left(\theta(0) \cdot \mu_t^{(0)} - \theta(1) \cdot \mu_t^{(1)}\right) dt, \\
\cdots & \quad \cdots, \\
d\mu_t^{(k)} &= \sqrt{2\mu_t^{(k)} \vee 0} \cdot db_t^{(k)} + \left(\theta(k-1) \cdot \mu_t^{(k-1)} - \theta(k) \cdot \mu_t^{(k)}\right) dt, \\
\cdots & \quad \cdots
\end{aligned}$$

where $b_t^{(0)}, b_t^{(1)}, \ldots, b_t^{(k)}, \ldots$ are mutually independent $BM^0(\mathbf{R})$'s.

4.2 Construction of a coalescing stochastic flow

The following application is due to Warren ([War 2]).

For $a, b, c \in \mathbf{R}$ such that $b \geq 0,\ a+b \geq 0,\ 0 \leq c \leq a+b$, define a transformation

$$h_{a,b,c}:\ [0,\infty) \ni x \mapsto h_{a,b,c}(x) \in [0,\infty)$$

by

$$h_{a,b,c}(x) = \begin{cases} x+a, & x > b \\ c, & 0 \leq x \leq b \end{cases}.$$

Then, $\mathcal{T} := \{h_{a,b,c};\ b \geq 0,\ a+b \geq 0,\ 0 \leq c \leq a+b\}$ forms a semigroup of transformations and the composition rule is given by

$$h_{a',b',c'} \circ h_{a,b,c} = h_{a",b",c"}, \quad a" = a + a',\ b" = b \vee (b'-a),\ c" = \begin{cases} c', & c \leq b' \\ c+a', & c > b' \end{cases}.$$

The topology of $\mathcal{T}$ is defined by the Euclidean topology of the parameter (a, b, c).

We consider the same joint process $(X^{(0)}, X^{(1)}, \cdots,)$ as in the previous subsection in which $\theta(k) = \theta > 0,\ k = 0, 1, \ldots$. Hence

$$X_t^{(0)} = B_t + L_t, \quad \text{where} \quad L_t = -\inf_{0 \leq s \leq t} B_s$$

and $(X^{(1)}, X^{(2)}, \cdots,)$ is uniquely determined (in the law sense) by the following SDE:

$$\begin{aligned}
X_t^{(1)} &= \int_0^t 1_{\{X_s^{(1)}>0\}} dB_s + \frac{\theta}{2}\int_0^t 1_{\{X_s^{(1)}=0\}} ds, \\
X_t^{(2)} &= \int_0^t 1_{\{X_s^{(1)}>0, X_s^{(2)}>0\}} dB_s + \frac{\theta}{2}\int_0^t 1_{\{X_s^{(1)}>0, X_s^{(2)}=0\}} ds, \\
\cdots & \quad \cdots, \\
X_t^{(k)} &= \int_0^t 1_{\{X_s^{(1)}>0,\ldots,X_s^{(k-1)}>0, X_s^{(k)}>0\}} dB_s + \frac{\theta}{2}\int_0^t 1_{\{X_s^{(1)}>0,\ldots,X_s^{(k-1)}>0, X_s^{(k)}=0\}} ds, \\
\cdots & \quad \cdots.
\end{aligned}$$

Define a family of $\mathcal{T}$-valued random variables $\phi_{s,t},\ 0 \leq s \leq t$, by

$$\phi_{s,t} = h_{a,b,c},\ a = B_t - B_s,\ b = -\inf_{s \leq u \leq t}(B_u - B_s),\ c = X_t^{(N_{s,t}+1)}$$

where $N_{s,t} = \inf_{s \leq u \leq t}\{N_u\}$.

Theorem 4.2. *([War 2]) The family of transformations $\{\phi_{s,t},\ 0 \leq s \leq t\}$ is a stochastic flow in the sense that*

(i) $\phi_{s,s} = \mathrm{id},\ \forall s.$

(ii) $\phi_{u,t} \circ \phi_{s,u} = \phi_{s,t},\ \forall s \leq u \leq t.$

(iii) If $0 \leq s_1 \leq s_2 \leq s_3 \leq \cdots$, then $\phi_{s_1,s_2}, \phi_{s_2,s_3}, \cdots$ are independent.

(iv) (stationarity) For $s \le t$ and $h > 0$, $\phi_{s,t} \stackrel{d}{=} \phi_{s+h,t+h}$.

(v) (continuity) For each $s \ge 0$, $[s,\infty) \ni t \mapsto \phi_{s,t} \in \mathcal{T}$ is continuous, a.s..

Obviously, the one-point motion $[s,\infty) \ni t \mapsto X_t := \phi_{s,t}(x)$, for each $x \in [0,\infty)$ and $s \ge 0$, is a reflecting Brownian motion on $[0,\infty)$ with a sticky boundary at 0 uniquely determined (in the law sense) by SDE

$$dX_t = 1_{\{X_t>0\}}dB_t + \frac{\theta}{2}1_{\{X_t=0\}}dt, \quad X_s = x.$$

If $\mathcal{F}_{s,t}$ is the σ-field generated by $\phi_{u,v}$, $s \le u \le v \le t$, then the family of σ-fields $\mathcal{F}_{s,t}$ generates a *predictable noise* in the sense of Tsirelson ([T]). A remarkable fact is that this noise is *not a Gaussian white noise*, that is, there is no Wiener process $W(t)$ (in any dimension) which can generate $\mathcal{F}_{s,t}$ as $\mathcal{F}_{s,t} = \sigma\{W(v) - W(u); s \le u \le v \le t\}$.

Remark 4.1. If we set

$$\mathcal{T}_1 = \{f_{a,b} := h_{a,b,0};\ b \ge 0,\ a+b \ge 0\}$$

and

$$\mathcal{T}_2 = \{g_{a,b} := h_{a,b,a+b};\ b \ge 0,\ a+b \ge 0\},$$

then $\mathcal{T}_1$ and $\mathcal{T}_2$ are algebraically isomorphic subgroups of $\mathcal{T}$ and, if we define two families of random transformations $\{\phi^{(1)}_{s,t},\ 0 \le s \le t\}$ and $\{\phi^{(2)}_{s,t},\ 0 \le s \le t\}$ by setting

$$\phi^{(1)}_{s,t} = f_{a,b}, \quad \phi^{(2)}_{s,t} = g_{a,b} \quad \text{where} \quad a = B_t - B_s,\ b = -\inf_{s \le u \le t}(B_u - B_s),$$

these families are stochastic flows which generate the same Gaussian white noise $\{\mathcal{F}_{s,t}\}$ given by $\mathcal{F}_{s,t} = \sigma\{B_v - B_u; s \le u \le v \le t\}$. One point motions are, for $\{\phi^{(1)}_{s,t}\}$, a Brownian motion on $[0,\infty)$ with an absorbing boundary (i.e. a trap) at 0 and, for $\{\phi^{(2)}_{s,t}\}$, a reflecting Brownian motion on $[0,\infty)$.

References

[BLL] J. Bertoin, J.-F. Le Gall and Y. Le Jan, Spatial branching processes and subordination, Can. J. Math.**49**(1997), 24-54

[B] P. Billigsley. *Convergence of Probability Measures,* Wiley, 1968

[Da] D. A. Dawson. Measure-valued Markov processes, École d'été de Probabilités de Saint-Flour, XXI-1991, **LNM 1541**, Springer(1993), 1-260.

[DS] J-S. Dhersin and L. Serlet. A stochastic calculus approach for the Brownian snake, preprint.

[Dy] E. B. Dynkin. Branching particle systems and superprocesses, Ann. Probab. **19**(1991), 1157-1194.

[F] W. Feller. Diffusion processes in genetics, *Proc. Second Berkeley Symp. Math. Statist. Prob.,* University of California Press(1951), Berkeley, 227-246.

[IW] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, 1989

[KW] K. Kawazu and S. Watanabe. Branching processes with immigration and related limit theorems, Teor. Veroyatnost. i. Primenen, **16**(1971), 34-51

[La] J. Lamperti. The limit of a sequence of a branching processes, Z. Wahrsch, verw. Gebiete **7**(1967), 271-288.

[Le 1] J. F. Le Gall. A class of path-valued Markov processes and its applications to superprocesses, Probab. Theory Relat. Fields **95**(1993), 24-46.

[Le 2] J. F. Le Gall. *Spatial Branching Processes, Random Snakes and Partial Differential Equations*, Lectures in Mathematics, ETH Zürich, Birkhäuser, 1999

[SW] T. Shiga and S. Watanabe. Bessel diffusions as a one-parameter family of diffusion processes, Z. Wahrsch, verw. Gebiete **27**(1973), 37-46.

[T] B. Tsirelson, Within and beyond the reach of Brownian innovation, to appear in *Proc. ICM, 1998.*

[War 1] J. Warren. Branching processes, the Ray-Knight theorem, and sticky Brownian motion, *Sém. Prob. XXXI*, **LNM 1655**, Springer(1997), 1-15.

[War 2] J. Warren. The noise made by a Poisson snake, preprint.

[Wat 1] S. Watanabe. Branching diffusions (superdiffusions) and random snakes, *Trends in Prob. Relat. Analysis*, Proc. SAP'96, World Scientific(1997), 289-304.

[Wat 2] S. Watanabe. Killing operations in super-diffusions by Brownian snakes, *Trends in Prob. Relat. Analysis*, Proc. SAP'98, World Scientific(1999), 177-190